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Percolation on Infinitely Ramified Fractals

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We present a family of exact fractals with a wide range of fractal and fracton dimensionalities. This includes the case of the fracton dimensionality of 2, which is critical for diffusion. This is achieved by adjusting the scaling factor as well as an internal geometrical parameter of the fractal. These fractals include the cases of finite and infinite ramification characterized by a ramification exponent ρ. The infinite ramification makes the problem of percolation on these lattices a nontrivial one. We give numerical evidence for a percolation transition on these fractals. This transition is tudied by a real-space renormalization group technique on lattices with fractal dimensionality $\bar{d}$ between 1 and 2. The critical exponents for percolation depend strongly on the geometry of the fractals.

KEY WORDS: Percolation; fractals; ramification phase transition; renormalization.

1. INTRODUCTION

Recently, there has been an increasing interest in exact mathematical fractals[1,2] and in phase transitions on fractal lattices.[2-4] The advantage of pure mathematical fractals is that one can generally calculate *exactly* the different critical exponents that characterize their various properties. This is important in view of the wide variety of physical systems which fractals seem to model.[1,2] Of particular interest is the question of the density of states of fractals[5-7] and its related exponent, the fracton dimensionality $\bar{\bar{d}}$.

In spite of the interest in phase transitions on fractal lattices, percolation, which is considered the most fundamental example of a transition,[8] has not yet been studied on fractals, probably because most of the research on phase transitions has been performed on Sierpinski gasketlike

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fractals which are finitely ramified. On these fractals, percolation takes place only in the uninteresting limit $p_c = 1$.

In this paper, we present a method to construct fractal lattices with any desired fractal and fracton dimensionality. These fractals have an infinite ramification whose critical exponent ρ will be shown to satisfy a particular inequality. Since ramification is infinite, the percolation problem in these fractals is a nontrivial one and $p_c < 1$.

The fact that $\bar{\bar{d}}$ can be adjusted is of interest because of its relation to the anomalous diffusion on fractal lattices and to their conductivity.[5,9–13] Particularly, $\bar{d} = 2$ is a critical dimension for anomalous diffusion in analogy with $d = 2$ for ordinary diffusion in homogeneous space. However, no fractal with $\bar{d} = 2$ has yet appeared in the literature.[13] We present numerical evidence for a percolation transition on the fractal lattices discussed above. The percolation problem is analyzed by the real-space renormalization group (RSRG) technique of Reynolds *et al.*[14] for fractals with fractal dimensionality $\bar{d}$ ranging from 1 to 2, thus providing a physical meaning to the analytic continuation of the percolation to $1 + \varepsilon$ dimensions.

2. THE FRACTAL FAMILY

We define a family of exact fractals as follows. One starts with a d-dimensional hypercube which is subdivided to b^d smaller hypercubes. For $d = 2$ the first stage of the fractal is a regular cartesian grid of $(b - bx)$ rows of b connected squares and bx rows each containing $(b - bx)$ squares. Each square belonging to the fractal is further diluted as in the first stage, and this procedure is continued indefinitely. In Fig. 1 we show an example of such a fractal embedded in a square with $b = 5$ and $x = 3/5$. Two stages of the procedure described above are shown. The figure is intended to show what

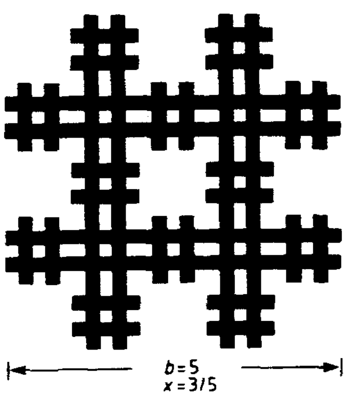

Fig. 1. Example of a fractal embedded in a square lattice with $b = 5$ and $x = 3/5$. Two iterations of the fractal are shown.

we mean by "a regular Cartesian grid...." Similarly for $d=3$ the first stage of the fractal is a regular Cartesian grid built of $(b-bx)$ planes of cubes arranged as in the first stage of the fractal for $d=2$, which are connected by the remaining bx planes each consisting of $(b-bx)^2$ cubes. So for d dimensions the first stage of the fractal consists of $b-bx$ hyperplanes which are just built as the first stage of the $d-1$ fractal and these planes are connected by bx hyperplanes each consisting of $(b-bx)^{d-1}$ hypercubes. Thus, the fractal dimensionality $\bar{d}$ is generally given by

$$b^{\bar{d}} = bx(b-bx)^{d-1} + (b-bx)b^{\overline{d-1}} \tag{1}$$

where $b^{\bar{1}} = b$ by definition. This recursion can be solved to yield

$$\begin{aligned} b^{\bar{d}} &= b^d f_d(x) = b^{d-\beta/\nu} \\ f_d(x) &= (1-x)^{d-1}[1+(d-1)x] = b^{-\beta/\nu} \end{aligned} \tag{2}$$

This relation for $\bar{d}$ can also be derived by counting the $d+1$ hypercubes which belong to each of the $[b(1-x)]^d$ "junctions" in the fractal, and then adding the $[b(1-x)]^{d-1}$ hypercubes for d of the remaining surface hyperplanes which were not counted. This shows that the fractals we defined are isotropic and do not have a preferred axis, as one might conclude from their description above.

In order to derive the conductivity exponent μ/ν, define the exponent $\bar{\zeta}$ by $r(ba) = b^{\bar{\zeta}} r(a)$ and the relation $\mu/\nu = d - 2 + \bar{\zeta}$.[9] We cannot calculate the recursion formula for $\bar{\zeta}$ exactly but in the following we shall present two different recursion formulae which give $\bar{\zeta}_1$ and $\bar{\zeta}_2$ such that $\zeta_1 \leqslant \bar{\zeta} \leqslant \bar{\zeta}_2$, and we shall discuss a limit for which $\bar{\zeta}_1 \to \bar{\zeta}_2$. We first present the arguments for $d=2$. The whole fractal has a resistance $r(a)$. Upon magnification by b, each of the $b^2-(bx)^2$ squares comprising the fractal has the same original resistance $r(a)$. The resistance of the whole fractal is now $r(ba)$.

Imagine, for example, that one measures conductivity of a fractal between two lines as in Fig. 2. In the first approach, one ignores the internal

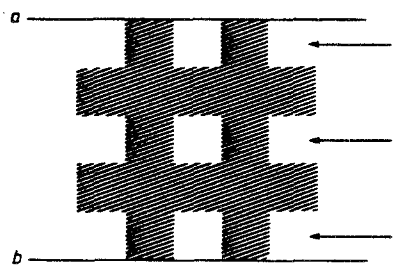

Fig. 2. Calculation of the conductivity. A voltage is applied between the two lines a and b. The arrows point to the rows contributing to the first term in Eq. (2).

structure of the squares in the succeeding stages and considers them as being "full" and not further diluted. Clearly this underestimates the real resistance $r(ba)$, since the different squares are assumed to match all along their boundaries, which is not in fact correct. By this assumption,

$$b^{\bar{\zeta}_1} = \frac{xb}{b - xb} + \frac{b - xb}{b} = \frac{x}{1 - x} + 1 - x, \qquad d = 2 \tag{3}$$

The first term represents the contribution of $(b - xb)$ parallel square resistance lying on xb different rows. These rows are marked with arrows in the example of Fig. 2. The second term represents the resistance of the remaining $(b - xb)$ rows which are in practice resistances with an area of b.

In the second approach, we nglect all the horizontal currents that might develop and thus only the contribution of $(b - bx)$ parallel chains of resistance $b\rho(a)$ contribute to $\rho(ba)$. Then,

$$b^{\bar{\zeta}_2} = \frac{b}{b - xb} = \frac{1}{1 - x}, \qquad d = 2 \tag{4}$$

This time we obviously overestimate the total resistance. Therefore, it follows that

$$\bar{\zeta}_1 \leqslant \bar{\zeta} \leqslant \bar{\zeta}_2 \tag{5}$$

This result is confirmed for the fractals in Fig. 1 and Fig. 4a by an exact enumeration of random walks diffusing on it. For the anomalous diffusion exponent of the fractal in Fig. 1, we find $D = 2.21 \pm 0.02$, so that $\bar{\zeta} = D - \bar{d} = 0.49 \pm 0.02$, which is within the range $(\bar{\zeta}_1, \bar{\zeta}_2) = (0.330, 1)$. For the fractal in Fig. 4a we find $D = 2.08 \pm 0.01$ so that $\bar{\zeta} = 0.19 \pm 0.01$, within the range $(\bar{\zeta}_1, \bar{\zeta}_2) = (0.14, 0.369)$. Note that in both cases $\bar{\zeta}$ is notably closer to $\bar{\zeta}_1$.

The above arguments can easily be extended for $d > 2$. One has

$$b^{\bar{\zeta}_1} = \frac{xb}{(b - xb)^{d-1}} + \frac{b - xb}{b^{d-1}}$$

or (6)

$$b^{d-2+\bar{\zeta}_1} = \frac{x}{(1 - x)^{d-1}} + \frac{1 - x}{f_{d-1}(x)} = b^{\mu_1/\nu}, \qquad d > 2$$

The first term represents the contribution of $(b - xb)^{d-1}$ parallel resistances lying on xb hyperplanes of dimension $d - 1$. The second term takes into account the remaining $b - xb$ hyperplanes whose resistance is inversely proportional to their area b^{d-1}. The generalization of the second approach is

$$b^{\bar{\zeta}_2} = \frac{b}{(b - xb)^{d-1}}$$

or (7)

$$b^{d-2+\bar{\zeta}_2} = \frac{1}{(1-x)^{d-1}} = b^{\mu_2/\nu}$$

The fracton dimensionality $\bar{\bar{d}}$ is related to the exponents discussed above[5] by

$$\bar{\bar{d}} = 2\bar{d}/D \tag{8}$$

where D is the anomalous diffusion exponent[10-12]:

$$D = 2 + \mu/\nu - \beta/\nu = \bar{d} + \bar{\zeta} \tag{9}$$

Thus, in fact, b and x determine $\bar{d}$ and $\bar{\bar{d}}$ of the fractal. In Table I, we present some numerical values for the exponents $\bar{d}$, $\bar{\bar{d}}$, and $\bar{\zeta}$ obtained for several

Table I. Dependence of the Exponents on b and x for $d = 3$.[a]

x	b	$\bar{d}$	$\bar{\bar{d}}$	$\bar{\zeta}$	ρ
0.5	2	2.00	1.66	0.42	0
			1.33	1.00	
	5	2.57	2.36	−0.39	1.14
			2.11	−0.14	
0.8	5	1.59	1.29	0.88	0
			1.23	1.00	
	20	2.24	1.99	0.01	0.93
			1.94	0.07	
	50	2.42	2.21	−0.23	1.18
			2.16	−0.18	
0.99	10^2	1.24	1.11	1.00	0
	10^4	2.12	2.00	0	1
	10^6	2.41	2.32	−0.33	1.33
0.999	10^3	1.16	1.07	1.00	0
	10^6	2.08	2.00	0	1
	10^9	2.39	2.32	−0.33	1.33

[a] Whenever there is a difference, the upper values refer to $\bar{\zeta}_1$ and the lower values refer to $\bar{\zeta}_2$.

values of b and x for $d=3$. For the exponents $\bar{\bar{d}}$ and $\bar{\zeta}$, the table gives two limiting values ($\bar{\zeta}_1$ and $\bar{\zeta}_2$) from Eqs. (6) and (7). However, for $x=1-\varepsilon$, these two values coincide with $O(\varepsilon)$ and then only one value is displayed in Table I. It is clearly seen that a wide range of values of $\bar{d}$ and $\bar{\bar{d}}$ can be achieved. This includes the cases of $\bar{\bar{d}}$ greater than, equal to, or smaller than 2. Moreover, it is seen from the table that more flexibility for $\bar{d}$ and $\bar{\bar{d}}$ is achieved in the limit of $x=1-\varepsilon$ with $\varepsilon \ll 1$, for which one knows the exact solution to $O(\varepsilon)$. As a matter of fact,

$$\bar{\bar{d}}=\frac{2\ln[b^d f_d(x)]}{\ln[b^2(1-x)^{1-d} f_d(x)]}+O(\varepsilon) \tag{10}$$

3. RAMIFICATION

We find it useful to use a definition of the ramification exponent as follows.[3] Suppose one can isolate a part of the fractal of linear size R by "cutting" it at the minimal number of places, $N_{\min}(R)$. Then as a consequence of self-similarity

$$N_{\min}(R) \sim R^{\rho} \tag{11}$$

The exponent ρ ranges between 0 for finitely ramified fractals (e.g., the Sierpinski gasket) to $\rho=d-1$ for homogeneous space. Moreover, let a given fractal have a fractal dimensionality $\bar{d}$ and a resistivity exponent $\bar{\zeta}$. Then in each section of the fractal of linear size R there are at least R fractal threads crossing it each being at most $R^{\bar{d}}/R^{\rho}$ long. Thus

$$R^{\bar{\zeta}} \leqslant R^{\bar{d}-2\rho} \qquad \text{or} \qquad \bar{\zeta} \leqslant \bar{d}-2\rho \tag{12}$$

But, since the anomalous diffusion exponent on any fractal satisfies $D=\bar{\zeta}+\bar{d} \geqslant 2$, then $\bar{\zeta} \geqslant 2-\bar{d}$ and Eq. (12) provides a better bound than the trivial one $\rho \leqslant \bar{d}-1 \leqslant d-1$. Note that ρ is defined by the *minimal* number of cuts one has to make in order to isolate a part of the fractal. If one replaces the requirement of the number of cuts being minimal by, let us say, the *mean* number of cuts needed, then one should get an averaged exponent ρ_{av} such that

$$\int_0^L R^{\rho_{\text{av}}}\, dR = L^{\rho_{\text{av}}+1} = L^{\bar{d}}$$

or (13)

$$\rho_{\text{av}} = \bar{d}-1$$

Then, clearly, $\rho \leqslant \rho_{av}$ and we must be cautions about minimizing $N_{min}(R)$. In fact, we could find $\rho' > \rho_{av}$ by making a number of cuts $N'(R)$ which increases with R faster than $R^{\rho_{av}}$. As an example, the fractal in Fig. 1 has $\rho = \ln 2/\ln 5 \simeq 0.43$, $\rho_{av} = \bar{d} - 1 \simeq 0.72$, and by cutting its central region on a continuous line lying on the shaded area we have $\rho' = 1$. We present the results for ρ on some exact fractals in Table I. For homogeneous space, $\rho = \rho_{av} = d - 1$, whereas for all of the connected fractals (exact and statistical) that we have checked, we find the inequality:

$$0 \leqslant \rho < \rho_{av} = \bar{d} - 1 \tag{14}$$

4. EVIDENCE FOR PERCOLATION ON FRACTALS

We expect percolation to take place at $p_c < 1$ whenever $\rho > 0$. The fractal displayed in Fig. 1 has $\rho = \ln 2/\ln 5$ so that nontrivial percolation may be expected. We carried out Monte Carlo simulations of site percolation on these fractal lattices built up to two, three, and four iterations. On each of these exact fractal lattices we grew clusters by the cluster growth method,[15] that is, a site near the center of the fractal was chosen as an origin and each around it belonging to the fractal were designated as being occupied with probability p or not with probability $1 - p$. The cluster growth was continued from the new cluster sites in a similar way till it either terminated or reached all the edges of the fractal. In Fig. 3 we show for each of the

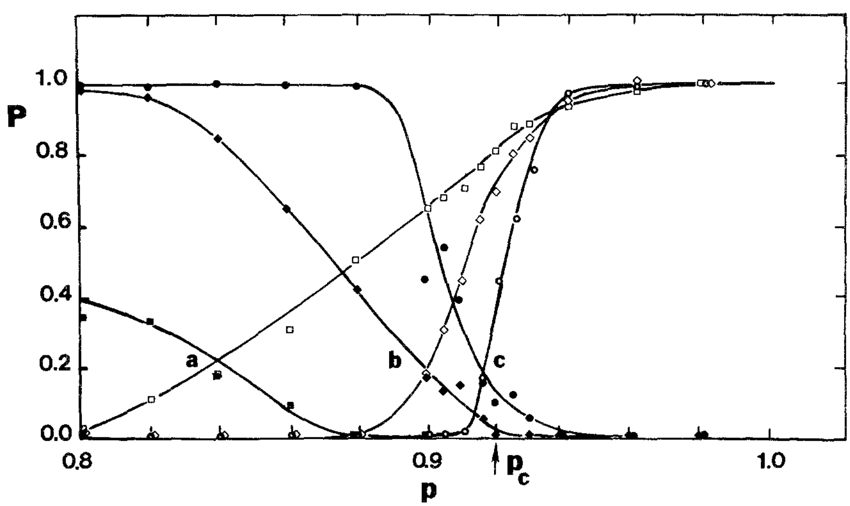

Fig. 3. Fraction of clusters reaching the edges of the lattice (white symbols) and of those which terminate before (black symbols), as a function of the concentration p for different lattice sizes: (a) 25 × 25, square symbols; (b) 125 × 125, diamond; (c) 625 × 625, circles.

fractal lattices described above (25×25, 125×125, and 625×625) the fraction of clusters reaching all the edges as well as the fraction of those which terminate before reaching any of the edges, as a function of the concentration p. The sum of the two fractions is not expected to be unity since there are percolation clusters which reach only part of the edges. The two curves intersect at points a, b, and c as shown in the figure. A sharper transition takes place as the lattice size increases. From the data for the largest lattice $p_c = 0.914 \pm 0.010$. Also extrapolation of the intersection points a, b, and c yields a percolation threshold $p_c \simeq 0.92$. We apply to this problem a RSRG approach of Reynolds *et al.*[14] A renormalized *fractal* cell is said to be occupied with probability p' if there exist a percolating cluster from the lower edge of the fractal to its upper one. Thus,

$$p' = p^{12} + 12p^{11}(1-p) + 51p^{10}(1-p)^2 + 96p^9(1-p)^3 + 96p^8(1-p)^4 + 46p^7(1-p)^5 + 14p^6(1-p)^6 + 2p^5(1-p)^7 \tag{15}$$

which has the trivial fixed points $p^* = 0, 1$ but also the nontrivial one $p_c \simeq 0.9221$ in good agreement with the numerical data.

5. PERCOLATION IN THE RSRG APPROACH

In order to investigate percolation as a function of the exponents $\bar{d}$ and of the exact fractal lattice we present the following fractal family. The first iteration of each member of the family is based on a ring embedded in a square of $b \times b$ sites. In Figs. 4a, 4b, and 4c we show the first iteration of the cases of $b = 2$, 3, and 4, respectively. Note that the case of $b = 2$ is just an homogeneous square lattice. The fractal dimensionality of a member of size b is

$$\bar{d} = \ln[4(b-1)]/\ln b \tag{16}$$

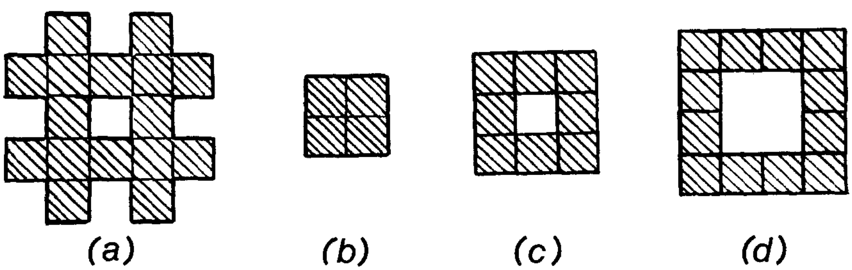

Fig. 4. (a), (b), and (c) are members of the fractal family (for $b = 2$, 3, and 4, respectively) studied by RSRG. In each case only the first iteration is shown.

and ρ is

$$\rho = \ln 2/\ln b \tag{17}$$

Thus, if we vary b, $\bar{d}$ ranges between 1 and 2 and ρ between 0 and 1. If one applies the same RSRG technique as before one obtains the general recursion formula

$$p' = 2p^b - p^{2b} \tag{18}$$

from which the percolation threshold p_c can be calculated. The correlation-length exponent is

$$\nu = \ln b/\ln[2b(1 - p_c^{b-1})] \tag{19}$$

The calculation of the exponent β is carried out by the ghost-site method (the ghost site is attached to each site on the cluster with a probability h). The bond between the ghost site and an occupied cell renormalizes to $h'p'$ if there is a percolating path connecting the lower edge of the cell and the ghost site. Thus one gets for $b = 2$ (homogeneous space)

$$h'p' = p^4 h_4 + 4p^3(1-p) h_3 \perp 2p^2(1-p)^2 h_2 \tag{20}$$

for $b = 3$

$$\begin{aligned} h'p' = {} & p^8 h_8 + 8p^7(1-p) h_7 + p^6(1-p)^2(14h_6 + 3h_5 + 2h_3) \\ & + p^5(1-p)^3(10h_5 + 7h_4 + 2h_3) + p^4(1-p)^4(6h_4 + 4h_3) \\ & + 2p^3(1-p)^5 h_3 \end{aligned} \tag{21}$$

and for $b = M \gg 1$

$$h'p' = p^{4M-4} h_{4M-4} + (4M-4) p^{4M-5}(1-p) h_{4M-5} + \cdots \tag{22}$$

where $h_l = 1 - (1-h)^l$. In Eq. (22) the missing terms make a negligible contribution. The exponent β is extracted by linearizing these equations at the critical point p_c [obtained from Eq. (18) and $h_c = 0$]. Then $\partial h'/\partial h|_{p_c, h_c} = \bar{d} - \beta/\nu$ so that using Eq. (4) and the value of ν found from Eq. (19) one is able to calculate β. Results for several values of b as well as for the limit $b = M \gg 1$ are displayed in Table II. According to this table p_c increases as $\bar{d}$ and ρ decrease toward the result obtained in one-dimensional space. The change of ρ and $\bar{d}$ also have a dramatic effect on the exponents ν and β of the percolation.

An important result of this work is the presentation of a physical model to the problem of the dimensionality $\bar{d}$ approaching to unity from

Table II. Characteristic Exponents of the Fractal Lattices and the Critical Exponents of Percolation on These Lattices

Fractal			Percolation		
b	$\bar{d}$	ρ	p_c	ν	β
2	2	1	0.6	1.63	0.63
3	1.89	0.63	0.85	2.13	0.27
4	1.79	0.5	0.92	2.43	
5	1.72	0.43	0.95	2.69	
10	1.56	0.30	0.989	4.60	
100	1.30	0.15	0.9999	6.79	
$M \gg 1$	$1+\frac{\ln 4}{\ln M}$	$\frac{\ln 2}{\ln M}$	$1-\frac{1}{M^2}$	$\frac{\ln M}{\ln 2}$	$\frac{\ln M}{M^2 \ln 2}$

above.[16,17] Table II implies that for $\bar{d} \to 1 (M \gg 1)$ the critical exponents are $\nu \sim 2/(\bar{d}-1)$, $\beta \sim (2/(\bar{d}-1)) \cdot \exp(-4 \ln 2/(\bar{d}-1))$ and $p_c \sim 1-\exp[-4 \ln 2/(\bar{d}-1(]$. It is interesting to note that these results have the same dependence on $d-1$ as those obtained by renormalization group technique. The constants, however, are different because of the specific geometric structure of the fractal family used.

6. SUMMARY AND DISCUSSION

We have presented a method for obtaining exact fractal lattices with any desired fractal dimensionality $\bar{d}$ and a fracton dimensionality $\bar{\bar{d}}$ to any degree of accuracy. We defined an exponent ρ which characterizes ramification and derived an inequality for it related to the resistivity exponent $\tilde{\zeta}$. The question of whether or not ρ is derivable from other critical exponents remains open. We have shown that a percolation transition takes place on exact fractal lattices with infinite ramification. The critical exponents of this percolation were estimated by a RSRG approach and found to have the same dependence on $\bar{d}-1=\varepsilon$ for $\bar{d}$ close to 1 as the one predicted for $d=1+\varepsilon$ by other renormalization techniques (which do not refer to fractals). Thus, the present work provides a possible physical meaning for the analytic continuation of the percolation problem to $d=1+\varepsilon$.

Finally, we note that an interesting situation arises for fractals with a fraction dimensionality $\bar{\bar{d}}_f$ ranging above and below 4/3 because a conjecture by Alexander and Orbach.[5] It is reasonable that $\bar{\bar{d}}_f$ of the *fractal* should be bigger or equal to $\bar{\bar{d}}_p$ of the *percolation* cluster resulting on it. Than for $\bar{\bar{d}}_f$ higher than 4/3 one expects the conjecture for percolation that $\bar{\bar{d}}_p = 4/3$ to hold. It would be interesting to have this conjecture checked for percolation on fractal lattices. Also the question of what happens for $\bar{\bar{d}}_f$ less than 4/3 is still unanswered. What would be the required value of $\bar{\bar{d}}_p$? There is still much numerical and theoretical work to be done on this intriguing subject.

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